

# Decay of the Remanent Magnetization in the Asymmetric Spin Chain

H. Rieger<sup>1,2</sup> and M. Schreckenberg<sup>1</sup>

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The dynamics of the one-dimensional spin glass with asymmetric interactions between neighboring spins is considered. We confine ourselves to discrete couplings with values  $\pm J$ . We show that the algebraic decay of the remanent magnetization of the infinite  $\pm J$ -spin chain at zero temperature is only valid for symmetric couplings. Our analytical investigations as well as computer simulations show stretched exponential decay for any finite concentration of antisymmetric bonds. Thus, the asymmetric  $\pm J$ -spin chain shows an asymmetry-induced phase transition at zero temperature.

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**KEY WORDS:** Ising model; spin glasses; relaxation phenomena; random matrices; multispin coding.

## 1. INTRODUCTION

Physical models of interacting spin systems have at least one feature in common: the symmetry of their interactions, which is due to Newton's law "*actio = reactio*." Since the variables of these models are microscopic objects, they have to obey this fundamental physical law. But if one considers larger objects, like neurons,<sup>(1)</sup> antibodies,<sup>(2)</sup> or populations (see ref. 3 and references therein), their interaction on a phenomenological level of description does not need to satisfy any physical law. For instance, the way one object within a system acts upon another object can be completely independent of the way the latter acts upon the first.

The application of statistical physics ideas to other branches of science, trying to model and calculate natural phenomena, has a long tradi-

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<sup>1</sup> Institut für Theoretische Physik, Universität zu Köln, 5000 Köln 41, Germany.

<sup>2</sup> Present address: Institute for Physical Science and Technology, University of Maryland, College Park, Maryland 20742.

tion. But in the last few years this expansion of theoretical physics (or penetration of biology, especially neurophysiology and immunology, computer science, or psychology, etc., into physics) becomes more and more obvious and is strongly correlated with the overwhelming use of modern computers in theoretical physics. For a review of, e.g., biologically motivated cellular automata see ref. 4 and references therein. Another well-known example is the so-called spin-glass model of neural networks,<sup>(1)</sup> and the immense work that has been undertaken in this research field is mainly concerned with the improvement of the original model<sup>(5)</sup> to develop and investigate a powerful device capable of associative memory, generalization, learning, parallel computing, retrieving sequences, and so on.

The aim of the present paper is more modest: motivated by the above activities and nonphysical models, we look for the possible consequences that can emerge if one ignores the restriction of symmetric interactions in (one-dimensional) spin systems. We do not want to speculate whether this plan could lead to a deeper understanding of neural networks. But the model considered here can be compared with, e.g., the spin chain with random field and its relevance for the theory of magnetism; and it is essentially the generalization of Glauber's model of a one-dimensional ferromagnet<sup>(6)</sup> to nonphysical interactions (using another language: a probabilistic cellular automaton with rules more general than those motivated by physics). Furthermore, it is a very interesting problem in nonequilibrium statistical physics to investigate the dynamical behavior of systems without the possibility of fulfilling any detailed balance condition and without the validity of such powerful tools like the fluctuation-dissipation theorem. In this context it should be mentioned that the asymmetric SK model,<sup>(7)</sup> which is the mean field version (i.e. in infinite dimensions) of the one-dimensional problem considered here, is still unsolved.<sup>(8-10)</sup> This was one more reason for us to consider related models in lower dimensions.

The central question for the asymmetric SK model is whether there is a spin-glass transition, i.e., a transition to a phase with broken ergodicity, if the couplings are not symmetric any more. In the disordered  $\pm J$ -spin chain one has no spin-glass behavior even for symmetric couplings, but one can nevertheless ask for changes in the relaxational behavior if asymmetry is present. At finite temperatures the relaxation of this model (of, e.g., the remanent magnetization) is always exponential. But at zero temperature there is a transition from algebraic to stretched exponential decay in the presence of asymmetric couplings, as was claimed in an earlier publication.<sup>(11)</sup> The present paper tries to prove this statement and to shed light on the interesting physics of this simple model.

The paper is organized as follows: In Section 2 the necessary definitions are given and known results are referred to as far as they are relevant

for the understanding of the analytical investigations. Section 3 presents the asymmetric  $\pm J$ -spin chain, and its connection to the bond-diluted spin chain is cleared up. Here also (i) the models with only one asymmetric bond are solved (the finite ring and the infinite chain with one spin fixed), as well as (ii) the periodic chain, where an antisymmetric bond occurs always after  $k$  symmetric bonds. These solvable cases give us a feeling for the disordered case and lead us to the hypothesis of a stretched exponential decay of the remanent magnetization.

The spectrum of the tridiagonal matrix describing the relaxation of the magnetization is investigated in Section 4. First we apply the transfer matrix technique and find the Lifshitz singularity at the band edge of the spectrum leading to stretched exponential decay. Then we compare the moments of the spectral density with those of a diluted spin chain (with  $\pm J$  interactions only) and demonstrate that the latter give an upper bound to the first. Since the diluted model shows a stretched exponential decay, it follows that the remanent magnetization of the asymmetric  $\pm J$ -spin chain cannot decay faster. In Section 5 results of computer simulations are presented, which are in full accordance with the theoretical predictions made in the previous sections. Furthermore, the simulations of the asymmetric  $\pm J$ -spin glass in two dimensions on the square lattice are shown. Conclusions and the outlook for forthcoming work<sup>(12)</sup> on the same kinds of models with external fields and continuous couplings instead of discrete ones considered here are given in Section 6. There are also two appendices containing some technical details.

## 2. DEFINITIONS AND KNOWN RESULTS

The model which we consider is defined as follows: The dynamical variables are  $N$  Ising spins  $\sigma_i$ , taking on the values  $+1$  and  $-1$  and interacting via a set of couplings  $J_{ij}$ . Each spin has a local field

$$h_i = b_i + \sum_{j(\neq i)} J_{ij} \sigma_j \quad (2.1)$$

Following Glauber,<sup>(6)</sup> we assume that the spins are coupled to a heat bath, leading to transition probabilities

$$w(\sigma_i \rightarrow \sigma'_i) = \frac{1}{2}(1 + \sigma'_i \tanh \beta h_i) \quad (2.2)$$

where  $\beta$  is the inverse temperature. Note that even in the case of vanishing temperature some noise is present, since for zero field  $h_i$  the spin flips stochastically with probability one-half. The stochastic process of the

movement of the system through its configuration space is described by a master equation for the probability distribution of spin configurations

$$\begin{aligned} \mathbf{P}\{\underline{\sigma}; t\} = & - \sum_{i=1}^N \{ \mathbf{w}(\sigma_i \rightarrow -\sigma_i) \mathbf{P}\{\underline{\sigma}; t\} - \mathbf{w}(-\sigma_i \rightarrow \sigma_i) \\ & \times \mathbf{P}\{(\sigma_1, \dots, -\sigma_i, \dots, \sigma_N); t\} \} \end{aligned} \quad (2.3)$$

Spatial correlations among the spins at equal times can be calculated via

$$\begin{aligned} \frac{\partial}{\partial t} \langle \sigma_{i_1} \cdots \sigma_{i_k} \rangle(t) = & \frac{\partial}{\partial t} \sum_{\underline{\sigma}} \sigma_{i_1} \cdots \sigma_{i_k} \mathbf{P}\{\underline{\sigma}; t\} \\ = & -2 \left\langle \sigma_{i_1} \cdots \sigma_{i_k} \sum_{l=1}^k \mathbf{w}(\sigma_{i_l} \rightarrow -\sigma_{i_l}) \right\rangle \end{aligned} \quad (2.4)$$

The reason why we consider only a one-dimensional system with nearest neighbor interactions in zero field ( $b_i=0$ ) is that these equations simplify considerably in this case. We have  $h_i = J_{i,i-1}\sigma_{i-1} + J_{i,i+1}\sigma_{i+1}$  (choosing periodic boundary conditions, we identify  $\sigma_{N+1}$  with  $\sigma_1$  and  $\sigma_0$  with  $\sigma_N$ ),

$$\tanh \beta h_i = \frac{1}{2} \{ \gamma_{i,i-1} \sigma_{i-1} + \gamma_{i,i+1} \sigma_{i+1} \} \quad (2.5)$$

with

$$\gamma_{i,i\pm 1} = \tanh \beta (J_{i,i+1} + J_{i,i-1}) \pm \tanh \beta (J_{i,i+1} - J_{i,i-1}) \quad (2.6)$$

Remembering the transition probabilities (2.2), one gets from (2.4)

$$\begin{aligned} \frac{\partial}{\partial t} \langle \sigma_{i_1} \cdots \sigma_{i_k} \rangle = & -k \langle \sigma_{i_1} \cdots \sigma_{i_k} \rangle \\ & + \frac{1}{2} \sum_{l=1}^k \langle \sigma_{i_1} \cdots \sigma_{i_{l-1}} (\gamma_{i_l, i_{l-1}} \sigma_{i_{l-1}} + \gamma_{i_l, i_{l+1}} \sigma_{i_{l+1}}) \sigma_{i_{l+1}} \cdots \sigma_{i_k} \rangle \end{aligned} \quad (2.7)$$

The case  $k=1$  yields the equation for the magnetization  $m_i(t) = \langle \sigma_i \rangle(t)$ :

$$\frac{\partial}{\partial t} m_i(t) = -m_i(t) + \frac{1}{2} \{ \gamma_{i,i-1} m_{i-1} + \gamma_{i,i+1} m_{i+1} \} \quad (2.8)$$

where according to the cyclic boundary condition we identify  $m_{N+1}$  with  $m_1$  and  $m_0$  with  $m_N$ .

Since we are mainly interested in the relaxational behavior of the magnetization, let us first clarify its relevance for the remaining correlation



tial with relaxation times smaller than  $(1-\gamma)^{-1}$ . Critical slowing down (indicating a transition to long-range order) occurs for the infinite system ( $N \rightarrow \infty$ ) at zero temperature. Consider the density of eigenvalues of  $\underline{\Gamma}$  (the spectral density),

$$\rho(\lambda) = \lim_{N \rightarrow \infty} \frac{1}{N} \left\{ \text{number of eigenvalues of } \underline{\Gamma} \text{ in the interval} \right. \\ \left. \times \left[ \lambda - \frac{d\lambda}{2}, \lambda + \frac{d\lambda}{2} \right] \right\} \quad (2.12)$$

where one gets from (2.11) for  $\gamma = 1$  ( $T = 0$ )

$$\rho(\lambda) = \frac{1}{\pi} \frac{1}{[1 - (\lambda + 1)^2]^{1/2}} \quad \text{for } \lambda \in [-2, 0] \quad (2.13)$$

and zero otherwise. Thus, one has an algebraic singularity at the zero eigenvalue:  $\rho(\lambda) \propto |\lambda|^{-1/2}$  for  $|\lambda| \ll 1$ , leading via Laplace transformation to an algebraic decay of the remanent magnetization:

$$M(t) = \frac{1}{N} \sum_{i=1}^N \langle \sigma_i(t) \sigma_i(0) \rangle \propto \frac{1}{\sqrt{t}} \quad \text{for } t \gg 1 \quad (2.14)$$

where  $\sigma_i(0)$  is a random initial configuration. Note that for an orthogonal diagonalizable matrix  $\underline{\Gamma}$  the remanent magnetization is indeed the Laplace transform of the spectral density:

$$M(t) = \int d\lambda \rho(\lambda) e^{\lambda t} \quad (2.15)$$

because a random initial configuration of the spins can be mapped onto a random superposition of eigenvectors of  $\underline{\Gamma}$  and to get the probability measure on the basis of eigenvectors, one just rotates the  $N$ -dimensional hypercube, representing the initial configuration of the spins, by an orthogonal transformation that diagonalizes  $\underline{\Gamma}$ .

The next case to be considered is the  $\pm J$  spin glass. Now,  $\gamma_{i,i+1} = \gamma_{i+1,i} = \pm \gamma/2$ , which means the off-diagonals of  $\underline{\Gamma}$  are occupied by  $+1$  and  $-1$  in a random manner. But apart from a frustration emerging from the cyclic boundary condition, nothing new can happen because all  $-1$  on the off-diagonal can be gauged away by a possible transformation  $\sigma_j \rightarrow -\sigma_j$ . Only if  $\text{sign}(\gamma_{12}\gamma_{23} \cdots \gamma_{N-1,N}\gamma_{N,1}) = -1$  is one left with one antiferromagnetic bond, leading to the absence of the eigenvalue  $\lambda = 0$ . The

spectral density is not altered by this possibility (since the limit  $N \rightarrow \infty$  is performed). Hence the remanent magnetization of the (symmetric)  $\pm J$ -spin chain decays also algebraically at zero temperature.

Before we come to the asymmetric case, we want to treat still another solvable model: the diluted one-dimensional ferromagnet (or the diluted  $\pm J$  spin glass, which is nearly the same, as we saw above). Here the bonds are removed randomly:

$$J_{i,i+1} = J_{i+1,i} = \begin{cases} J & \text{with probability } p \\ 0 & \text{with probability } 1 - p \end{cases} \quad (2.16)$$

Note that the infinite chain splits into finite ferromagnetic chains of length  $l$  with free boundaries. The probability of a randomly chosen spin to belong to such a segment  $l$  is  $p_l = l(1-p)^2 p^{l-1}$  and the remanent magnetization of the diluted ferromagnetic chain is given by the remanent magnetization of these finite chains weighted with their probabilities:

$$M(t) = \sum_{l=1}^{\infty} l(1-p)^2 p^{l-2} M_l(t) \quad (2.17)$$

Since

$$M_l(t) \propto \exp t \cdot \left( -1 + \cos \frac{\pi}{l+1} \right)$$

for zero temperature and long times  $t$ , one can evaluate the asymptotic behavior of the sum (2.17) by approximating it through an integral and applying the method of steepest descent, resulting in stretched exponential decay with exponent  $1/3$ ,<sup>(13-15),3</sup>

$$M(t) \propto \exp \left( - \frac{t^{1/3}}{\tau(p)} \right) \quad \text{for } t \gg \tau(p) \quad (2.18)$$

where  $\tau(p)$  is a time scale depending on the dilution. For smaller time scales than  $\tau(p)$  the relaxational behavior is different.

Thus, we have a phase transition at zero temperature from algebraic to stretched exponential decay that is induced by the dilution of the chain and which occurs at  $p = 1$ . We emphasize this fact since there are certain similarities between the diluted chain and the asymmetric spin chain, as will be explained below.

<sup>3</sup> For a related work on stretched exponential decay in disordered Ising systems in general dimensions see ref. 16.

### 3. THE ASYMMETRIC $\pm J$ -SPIN CHAIN

By randomly introducing antisymmetric bond pairs  $J_{i,i+1} = -J_{i+1,i}$  into a  $\pm J$ -spin chain, one gets the asymmetric  $\pm J$ -spin chain, i.e.,  $J_{i,i\pm 1} = \pm J$ ,  $J_{i,i+1} = J_{i+1,i}$  with probability  $p$  and  $J_{i,i+1} = -J_{i+1,i}$  with probability  $1 - p$ . The tridiagonal matrix  $\underline{\Gamma} = -\underline{I} + (\gamma/2)\underline{A}$  that has to be diagonalized to solve the system (2.8) is now no longer symmetric and  $\lambda$  is given by

$$\underline{A} = \begin{pmatrix} 0 & \theta_1^+ & & & & \theta_N^- \\ \theta_1^- & 0 & \theta_2^+ & & \circ & \\ & \theta_2^- & 0 & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & \circ & & \ddots & & 0 & \theta_{N-1}^+ \\ \theta_N^+ & & & & & \theta_{N-1}^- & 0 \end{pmatrix} \quad (3.1)$$

where  $\theta_i^\pm \in \{-1, +1\}$  and

$$\theta_i^+ \cdot \theta_i^- = \begin{cases} +1 & \text{with probability } p \\ -1 & \text{with probability } 1 - p \end{cases} \quad (3.2)$$

To get rid of some superfluous signs, one can introduce new spin variables and new fields

$$\tilde{\sigma}_i = J_{12}J_{23} \cdots J_{i-1,i} \sigma_i \quad \text{and} \quad \tilde{h}_i = J_{12}J_{23} \cdots J_{i-1,i} h_i \quad (3.3)$$

by which the matrix to be diagonalized becomes

$$\tilde{\underline{A}} = \begin{pmatrix} 0 & 1 & & & & \theta_N \\ \theta_1 & 0 & 1 & \circ & & \\ & \theta_2 & 0 & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & \circ & & \ddots & & 0 & 1 \\ 1 & & & & & \theta_{N-1} & 0 \end{pmatrix} \quad \text{with } \theta_i = \theta_i^- \theta_i^+ \quad (3.4)$$

[Here it is assumed that  $\text{sign}(J_{12}J_{23} \cdots J_{N-1,N}J_{N1}) = +1$ .] This means that the asymmetric spin chain is equivalent to a chain where each spin has a positive coupling coming from its right neighbor and a coupling that is positive or negative with probability  $p$  or  $1 - p$  coming from its left neighbor. This formulation is most advantageous for computer simulations and will be used in Section 5.



As one can see from (3.4), the matrix is not symmetric any more, but can be symmetrized by using a similarity transformation resulting in a complex symmetric matrix  $\underline{\underline{A}}$  with

$$\underline{\underline{A}} = \begin{pmatrix} 0 & \bar{\theta}_1 & & & & & & & \bar{\theta}_N \\ \bar{\theta}_1 & 0 & \bar{\theta}_2 & & & & \circ & & \\ & \bar{\theta}_2 & 0 & \ddots & & & & & \\ & & \ddots & \ddots & \ddots & & & & \\ & & & \ddots & \ddots & & & & \\ \circ & & & & & & & & \\ & & & & & & 0 & & \bar{\theta}_{N-1} \\ \bar{\theta}_N & & & & & & \bar{\theta}_{N-1} & & 0 \end{pmatrix} \quad \text{with } \bar{\theta}_k = (\theta_i^- \theta_i^+)^{1/2} \quad (3.5)$$

There is no way out of the fact that  $\underline{\underline{A}}$  needs not to be diagonalizable. Still (because of the symmetry of  $\underline{\underline{A}}$ ) the eigenvectors corresponding to different eigenvalues are orthogonal and linear independent eigenvectors corresponding to multiple eigenvalues can be orthogonalized, but the algebraic and geometric multiplicity of eigenvalues needs not to be the same any more. If we arbitrarily exclude these cases by restricting the ensemble of random matrices, we can suppose that  $\underline{\underline{A}}$  is diagonalizable by an orthogonal transformation. We shall see later that these excluded cases in fact do not occur very often, but it needs to be clarified whether the restricted ensemble has full measure or whether the degenerate cases have any qualitative consequences for the results.

The first thing one can learn about the spectrum of  $\underline{\underline{A}}$  (respectively  $\bar{\underline{\underline{A}}}$ ) is the following: From linear algebra one knows that

$$\text{Max}\{\text{Re } \lambda \mid \lambda \text{ eigenvalue of } \bar{\underline{\underline{A}}}\} \leq \text{Max}\{\lambda \mid \lambda \text{ eigenvalue of Re } \underline{\underline{A}}\} \quad (3.6)$$

where  $\text{Re } \bar{\underline{\underline{A}}}$  means the real part of the matrix  $\bar{\underline{\underline{A}}}$ . This real symmetric matrix  $\text{Re } \bar{\underline{\underline{A}}}$  looks similar to the matrix of the corresponding diluted chain, where each antisymmetric bond pair is replaced by a removed bond pair.

The similarity is not complete, because the free boundary conditions of the finite segments of the bond-diluted chain would imply a two times larger element in the first and last rows of the corresponding block matrix (leading to an eigenvalue zero, as it should be). This is not the case for  $\text{Re } \bar{\underline{\underline{A}}}$ . The isolated  $l \times l$  block matrices of  $\text{Re } \bar{\underline{\underline{A}}}$  with the same elements in the first and last rows as in the others occur when one considers a finite ferromagnetic chain of length  $l$ , where the first and the last spins are coupled to the left, respectively right, with a “chaotic” spin flipping randomly in time and thus having a zero magnetization. This picture is even more adequate for the asymmetric chain, since an antisymmetric bond pair produces this chaos, as we shall see later. The difference between the

spectrum of both models, the one with free boundaries and the other with chaotic spin at the boundaries, is not quite large (as one can easily calculate), so that we shall call  $\text{Re } \underline{\underline{A}}$  the matrix of the corresponding diluted model.

If within the asymmetric chain the longest segment of spins interacting via symmetric bond pairs only has length  $l_{\max}$ , the inequality (3.6) implies that

$$\text{Max}\{\text{Re } \lambda \mid \lambda \text{ eigenvalue of } \underline{\underline{A}}\} < \cos \frac{\pi}{l_{\max} + 2} \quad (3.7)$$

In other words, the remanent magnetization of a (finite or infinite) asymmetric spin chain with not more than  $l_{\max}$  symmetric bond pairs in succession decays exponentially with a relaxation time smaller than  $(1 - \cos[\pi/(l_{\max} + 2)])^{-1}$ . This can also be seen in the following examples.

### 3.1. One Antisymmetric Bond

Consider the finite ring first. Let all  $\theta_i = 1$  [see (3.4)] except  $\theta_N = -1$ . Then the impurity, i.e., the antisymmetric bond pair, is located between  $\sigma_N$  and  $\sigma_1$ . The eigenvalues of  $\underline{\underline{A}}$  are given by

$$\lambda^{(n)} = \cos \frac{\pi n}{N} \quad (n = 1, \dots, N) \quad \text{and} \quad \lambda = 0 \quad (3.8)$$

For  $N$  even, the root  $\lambda = 0$  of the secular equation of  $\underline{\underline{A}}$  is twofold, but there exists only one zero-eigenvector of the matrix  $\underline{\underline{A}}$ . This is an example of the above-mentioned degeneracy of the matrix  $\underline{\underline{A}}$ . The eigenvectors are given by

$$m_i^{(n)} = \begin{cases} \sin \frac{\pi n i}{N} & \text{for } n \text{ even} \\ \sin \frac{\pi n (i-1)}{N} & \text{for } n \text{ odd} \end{cases} \quad (3.9)$$

Furthermore, one observes (because the eigenvalues of  $\text{Re } \underline{\underline{A}}$  are  $\lambda^{(n)} = \cos[\pi n/(N+1)]$ ,  $n = 1, \dots, N$ ) that all eigenvalues of  $\underline{\underline{A}}$  are smaller than  $\text{Max}\{\lambda \mid \lambda \text{ eigenvalue of } \text{Re } \underline{\underline{A}}\} = \cos[1/(N+1)]$ , i.e., smaller than the largest eigenvalue of the corresponding diluted chain. Finally, it can be seen that the remanent magnetization decays completely to zero, whereas in the finite ferromagnetic ring it decays with probability one to a non-vanishing value. This indicates that the influence of the impurity spreads through the whole chain after long enough time.

This epidemic effect of a single impurity can also be seen in the following situation: Consider a half-infinite chain, where the first spin is fixed ( $\sigma_0 = +1$ ) and all bonds are symmetric and equal to  $+1$  except one antisymmetric bond pair placed  $k$  bonds away from the fixed spin (i.e.,  $J_{k,k+1} = -J$  for one  $k \geq 2$  and all other  $J_{i,i\pm 1} = +J$ ; see Fig. 1). Then the stationary values of the magnetizations are given by

$$m_i = \begin{cases} 1 - \frac{2i}{2k+1} & i = 1, \dots, k \\ \frac{1}{2k+1} & i > k \end{cases} \tag{3.10}$$

If all bonds were ferromagnetic, this chain would become fully magnetized (i.e.,  $m_i = 1$  for all sites  $i$ ). The presence of one antisymmetric bond pair reduces this magnetization to a much smaller value (depending on the distance between the fixed spin and the impurity) within the whole chain.

This behavior can be explained by the following observation. Look at two spins  $\sigma_i$  and  $\sigma_{i+1}$  to the left and right of an antisymmetric bond pair  $J_{i,i+1} = -1, J_{i+1,i} = +1$ . Suppose the left spin  $\sigma_i$  points upward. If the right spin  $\sigma_{i+1}$  points downward, it has either zero or positive field (depending on its right neighbor  $\sigma_{i+2}$ ). Hence, it flips at least with probability one-half. The only chance for it not to flip is to point upward, too. But in this case the left spin has either zero or positive field (depending on its left neighbor  $\sigma_{i-1}$ ), which means that now this spin flips at least with probability one-half. Thus, in all possible situations an antisymmetric bond pair is a source

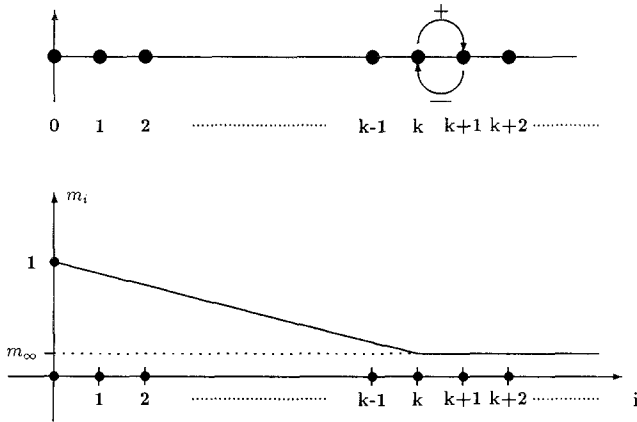


Fig. 1. The half-infinite spin chain. The leftmost spin is fixed; a single antisymmetric bond pair is located between  $\sigma_k$  and  $\sigma_{k+1}$ , as indicated above. The resulting values for the steady state of the magnetizations are depicted below.

of complete randomness within the chain since one of the two neighboring spins is always flipping in a random manner.

Considering, for example, an initially fully magnetized ferromagnetic spin chain with only one antisymmetric bond pair, this impurity produces disturbances of flipped spins that propagate like a random walk in one dimension through the whole chain. If this chain is finite of length  $l$  (or one looks at finite segments, bounded by two antisymmetric bonds, within a infinite chain), this process destroys the magnetization after a time of the order  $l^2$ . This is also the order of magnitude of the relaxation time of a finite chain or isolated segment of length  $l$  consisting only of symmetric bonds. These reflections lead us to detect the analogy of the relaxational behavior of the asymmetric chain and the corresponding diluted chain:

For the sake of simplicity, we suppose that initially all spins are pointing upward. The asymmetric spin chain is composed of finite segments of purely ferromagnetic bonds bounded to the left and right by antisymmetric bond pairs, which—as we mentioned above—produce random spin flips propagating randomly through the fully magnetized segments. Thus, the magnetization of a segment of length  $l$  is destroyed after a characteristic time proportional to  $l^2$ , but the probability of a spin to be a member of such a segment is  $l(1-p)^2 p^{l-1}$ . Hence, the magnetization is approximately given by

$$M(t) \approx \sum_{l=1}^{\infty} l(1-p)^2 p^{l-1} \exp(-t/cl^2) \tag{3.11}$$

which results in a stretched exponential decay for large enough time, in complete analogy to the diluted chain [see (2.17), (2.18)].

### 3.2. Periodic Chain

Consider now a spin chain where an antisymmetric bond pair occurs periodically after  $k$  ferromagnetic bond pairs, i.e.,

$$J_{i,i+1} = \begin{cases} -J_{i+1,i} & \text{if } i = k + 1, 2k + 2, 3k + 3, \dots \\ +J_{i+1,i} & \text{otherwise} \end{cases} \tag{3.12}$$

Defining the matrix

$$\underline{A}^{(n)} = \begin{pmatrix} 0 & \theta_1^+ & & & & \\ \theta_1^- & 0 & \theta_2^+ & & \circ & \\ & \theta_2^- & 0 & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ \circ & & & \ddots & 0 & \theta_{n-1}^+ \\ & & & & \theta_{n-1}^- & 0 \end{pmatrix} \tag{3.13}$$

[cf. (3.1)], one observes for the determinants  $D^{(n)} := \det(\underline{A}^{(n)} + \underline{I}^{(n)})$ , where  $\underline{I}^{(n)}$  is the  $(n + 1) \times (n + 1)$  unity matrix, the following recursion relation:

$$D^{(n+1)} = \lambda D^{(n)} - \theta_n D^{(n-1)}, \quad \text{with } D^{(0)} = 1, \quad D^{(1)} = \lambda \quad (3.14)$$

In this case one has  $\theta_n = \theta_n^+ \theta_n^- = -1$  for  $n = k + 1, 2k + 2, \dots$  and  $\theta_n = +1$  otherwise. The zeros of  $D^{(N-1)}(\lambda)$  are the eigenvalues of  $\underline{A}$ . Furthermore, note that the  $n$ th approximants of a continued fraction obey<sup>(17)</sup>

$$S^{(n)} = \frac{1}{\lambda - \frac{\theta_1}{\lambda - \frac{\theta_2}{\lambda - \dots - \frac{\theta_n}{\lambda}}}} = \frac{A_n(\lambda)}{B_n(\lambda)} \quad (3.15)$$

where  $A_n(\lambda)$  and  $B_n(\lambda)$  are polynomials of degree  $n$  in  $\lambda$  and the denominator satisfies the same recursion relation as the determinants  $D^{(n)}(\lambda)$ , i.e.,

$$D^{(N-1)}(\lambda) = B_{N-1}(\lambda) \quad (3.16)$$

Since one knows how to calculate the value of a periodic continued fraction, one can get information about the distribution of eigenvalues of the matrix  $\lambda$  in the limit  $N \rightarrow \infty$  via the singularities of  $S(\lambda) = \lim_{N \rightarrow \infty} S^{(N)}(\lambda)$ . In Appendix A we show that

$$x \cdot S(x) = -\frac{b_k(x)}{2} + \left[ \frac{b_k^2(x)}{4} + x^2 \right]^2 \quad (3.17)$$

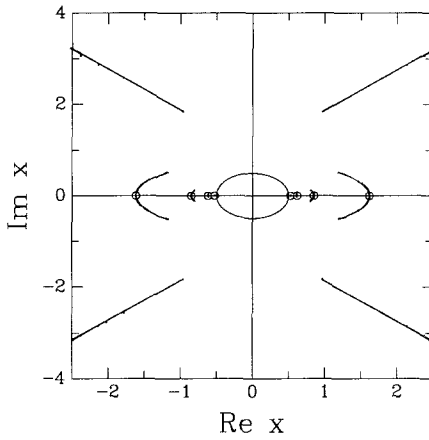
with

$$b_k(x) = (\rho_+ - \rho_-) \frac{\rho_+^k + \rho_-^k}{\rho_+^k - \rho_-^k}, \quad \rho_{\pm} = \frac{1}{2} \pm \left( \frac{1}{4} - x^2 \right)^2 \quad (3.18)$$

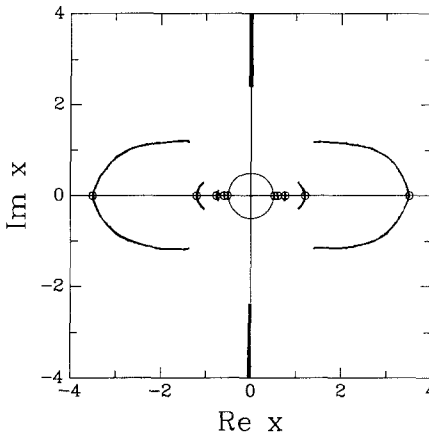
and  $x = \lambda^{-1}$ . (Remember that to get the eigenvalues of  $\underline{\Gamma}$  itself one has to multiply  $\lambda$  with  $\gamma/2$  and shift the result by  $-1$ .) The poles of  $b_k(x)$  are the solutions of  $\rho_+^k = \rho_-^k$ , which are  $x = \pm [2 \cos(\pi n/k)]^{-1}$ ,  $n = 1, \dots, k$ . This means for the Laplace transform  $M(z) = \int_0^\infty dt \exp(-zt) M(t)$  of the remanent magnetization that it has poles at

$$z_n = -1 \pm \frac{\gamma}{2} \cos \frac{\pi n}{k}, \quad n = 1, \dots, k - 1 \quad (3.19)$$

and no other poles (except  $z = -1$ ). But there are other algebraic singularities [of the form  $(z - z_0)^{1/2}$  near certain values  $z_0$ ] emerging from the cuts of the square root in  $S(x)$ . Therefore we have to look for the values of  $x$  where  $b_k^2(x) + 4x^2 = 0$ . Because there is no analytical solution of this equation for general  $k$ , we investigated the  $2k$  roots of this equation numerically; see Fig. 2: Two cuts belong to one pole, one of them leaving



(a)



(b)

Fig. 2. The arrangement of the poles and algebraic cuts of the continued fraction (3.15) within the complex plane of the variable  $x$ . (a)  $k = 10$  (even); (b)  $k = 11$  (odd). The circle of radius one-half and center zero indicates a part of the domain of analyticity (in the variable  $x$ ) of the continued fraction (3.15). The point  $x = 1/2$  corresponds to the zero eigenvalue of the matrix  $\underline{L}$ . For further explanation see text.

into the upper complex half-plane, the other into the lower half-plane, and terminate in the above-mentioned algebraic singularities, where the argument of the square root becomes zero. There are four (if  $k$  is even) or two (if  $k$  is odd) cuts belonging to the poles of  $b_k(x)$  at infinity. One observes that the density of the singularities in the vicinity of  $x = 1/2$  [corresponding to  $z = 0$ , which determines the asymptotic behavior of  $M(t)$ ] is not altered in an essential way by the presence of these cuts.

Thus one concludes that the asymptotic behavior of the remanent magnetization is similar to that having poles in its Laplace transform located at  $z_n$ ; see (3.19). But these can be compared with the eigenvalues of the matrix  $\underline{\Gamma}$  of the corresponding periodically diluted chain that is simply identical to one finite chain of length  $k + 1$  with “chaotic” spin at the boundaries. Since the eigenvalues of the latter are

$$\lambda^n = -1 + \frac{\gamma}{2} \cos \frac{\pi n}{k+2}, \quad n = 1, \dots, k+1$$

we observe that the remanent magnetization of the periodic asymmetric chain decays faster than the corresponding diluted chain.

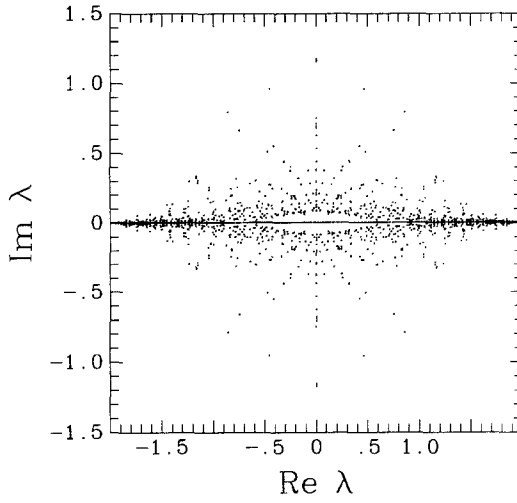
#### 4. INVESTIGATION OF THE SPECTRUM

As we mentioned earlier, we expect a stretched exponential decay of the remanent magnetization of the asymmetric  $\pm J$ -spin chain at zero temperature, since we detected the analogy to a ferromagnetic spin chain with randomly distributed “chaotic” spins, which we have called the corresponding diluted model. In this section we intend to test this expectation analytically. As we mentioned in the beginning of the previous section, we assume that  $\underline{A}$  is diagonalizable and hence the remanent magnetization is given by

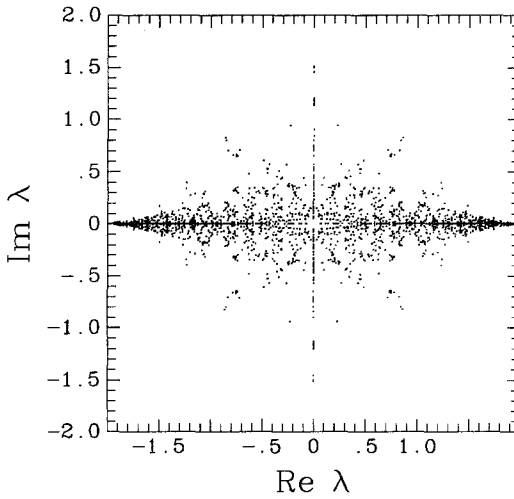
$$M(t) = \int_{\mathbb{C}} d^2\lambda e^{\lambda t} \rho(\lambda) \tag{4.1}$$

where  $\rho(\lambda)$  is the spectral density of the matrix  $\underline{\Gamma}$  [see text before (3.1)]. Therefore we investigate in this section the density of eigenvalues of the matrix  $\underline{A}$  (3.1), especially its behavior in the vicinity of the point  $\lambda = 2$ . This point corresponds to the value  $\lambda = 0$  within the spectrum of the matrix  $\underline{\Gamma}$  and the density of eigenvalues there determines the asymptotic behavior of  $M(t)$ .

To get some insight into the complexity of the problem look at Fig. 3, where we plotted within the complex plane the eigenvalues of several random matrices  $\underline{A}$  for fixed amount  $p$  of symmetric bond pairs. As we know,



(a)



(b)

Fig. 3. The spectrum of samples of tridiagonal matrices  $\underline{A}$  (3.1) for different values of  $p$ : (a)  $p=0.95$ , (b)  $p=0.90$ , (c)  $p=0.85$ . Each point corresponds to one eigenvalue of an  $N \times N$  matrix;  $N$  is 500 and for each picture ten matrices have been diagonalized numerically.



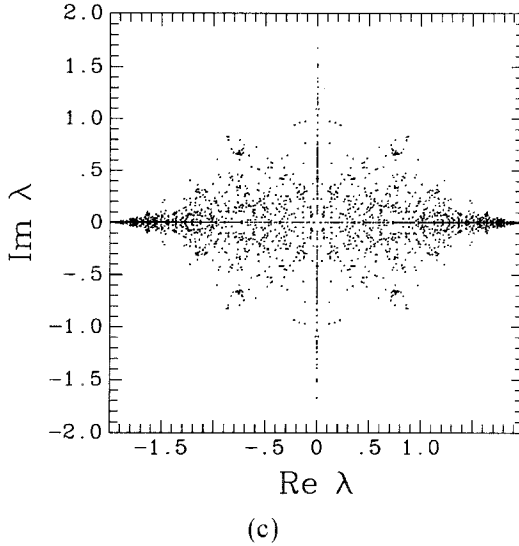


Fig. 3. (Continued)

for  $p = 1$  the spectrum of  $\underline{A}$  is contained in the interval  $[-2, 2]$  of the real axis with diverging density at the band edges  $-2$  and  $+2$ . For  $p < 1$  the spectrum explodes into the upper and lower half-planes and forms extremely complicated structures. We did not gather enough numerical data to give detailed information about, e.g., the support of the spectrum, but we expect it to be a fractal (as in disordered binary alloys<sup>(18,19)</sup>) and perhaps future work will also detect a connection to Julia sets<sup>(20)</sup>, because, as we shall see later, an equivalent formulation of the eigenvalue problem makes use of iterated Möbius transformations.

The first thing one is tempted to do if one wants to make some statements about the spectrum of tridiagonal matrices is to apply Dyson's formalism,<sup>(21)</sup> which works (in principle) in our case, because the elements are not correlated [as is most obvious in the representation (3.4)]. But as was mentioned already in an earlier publication,<sup>(11)</sup> this formulation leads to a functional equation that is not solvable for this kind of distribution of the random variables. This should not have been expected, since almost all distributions, especially discrete ones, within the field of disordered one-dimensional systems lead to these difficulties. There seems also to be no approximation technique that would be able to handle this equation, at least in the vicinity of the most interesting eigenvalue  $\lambda = 2$  of  $\underline{A}$ .

Our next step is to investigate the spectrum by the transfer matrix method and by the method of moments.

### 4.1. Transfer Matrix Method

The eigenvalues of the matrix  $\underline{A}$  (3.1) are determined—via a similarity transformation to the matrix (3.4)—by the eigenvalue equation

$$\lambda m_i = \theta_i m_{i-1} + m_{i+1} \tag{4.2}$$

where  $\theta_i = +1$  with probability  $p$  and  $\theta_i = -1$  with probability  $1 - p$ . Equation (4.2) can be expressed with the help of transfer matrices:

$$\begin{pmatrix} m_{i+1} \\ m_i \end{pmatrix} = \underbrace{\begin{pmatrix} \lambda & -\theta_i \\ 1 & 0 \end{pmatrix}}_{=: T_{\theta_i}} \begin{pmatrix} m_i \\ m_{i-1} \end{pmatrix}, \quad m_0 = m_{N+1} = 0 \tag{4.3}$$

The boundary conditions do not influence the distribution of eigenvalues in the limit  $N \rightarrow \infty$ . Choose  $\beta$  in such a way that  $\lambda = 2 \cos \beta$  ( $\text{Re } \beta \in [0, \pi[$ ); the eigenvalues of  $T_+$  are given by  $x = \exp(\pm i\beta)$ . If we concentrate on real eigenvalues, then  $\lambda \in [-2, 2]$  and hence  $\beta \in [0, \pi[$ . It is convenient<sup>(19,22)</sup> to perform a transformation that diagonalizes  $T_+$ :

$$\mathcal{T}_+^{-1} = U^{-1} T_+^{-1} U = \begin{pmatrix} e^{-i\beta} & 0 \\ 0 & e^{i\beta} \end{pmatrix}, \quad U = \frac{1}{(2i \sin \beta)^{1/2}} \begin{pmatrix} 1 & -1 \\ e^{-i\beta} & -e^{i\beta} \end{pmatrix} \tag{4.4}$$

Hence, each symmetric bond pair yields a simple transformation of the vector  $U(m_i, m_{i-1})^T$ , whereas an antisymmetric bond pair results in the following transfer matrix:

$$\mathcal{T}_-^{-1} = U^{-1} T_-^{-1} U = \frac{1}{i \sin \delta} \begin{pmatrix} \cos \delta e^{-i\beta} & -e^{i\beta + i\delta} \\ e^{-i\beta - i\delta} & -\cos \delta e^{i\beta} \end{pmatrix} =: Q_\beta \tag{4.5}$$

where we defined  $\delta = \beta$  because then  $Q_\beta (\mathcal{T}_+^{-1})^{j-1} = Q_{j\beta}$ . With  $(b_n^+, b_n^-)^T := U^{-1}(m_{n+1}, m_n)^T$ , the eigenvalue problem can be formulated as follows:

$$\begin{pmatrix} b_{n-1}^+ \\ b_{n-1}^- \end{pmatrix} = \mathcal{T}_n^{-1} \mathcal{T}_{n+1}^{-1} \dots \mathcal{T}_N^{-1} c \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad c = \frac{-m_N}{(2i \sin \beta)^{1/2}} \tag{4.6}$$

with  $(b_0^+, b_0^-)^T = \tilde{c}(e^{i\beta})^T$ ,  $\tilde{c} = m_1 / (2i \sin \beta)^{1/2}$ . It is sufficient to consider the quotient  $z_n := b_n^+ / b_n^-$ . Then the transformations  $z_n \rightarrow z_{n-1}$  become Möbius transformations:

$$z_{n-1} = \begin{cases} \exp(-2i\beta) \cdot z_n & \text{if } \theta_n = +1 \text{ (symmetric bond)} \\ -\frac{az_n + b}{b^*z_n + a^*} =: R_\beta(z_n) & \text{if } \theta_n = -1 \text{ (antisymmetric bond)} \end{cases} \tag{4.7}$$

starting at  $z_N = 1$  and ending at  $z_0 = e^{2i\beta}$ . The above formula is only fully correct for real  $\delta$ , i.e., real eigenvalues  $\lambda$ , since  $a = \cos \delta e^{-i\beta}$  and  $b = -e^{i\beta + i\delta}$ . If  $\delta$  has a nonvanishing imaginary part, it is  $\cos \delta \neq (\cos \delta)^*$ , and  $a^*$  within the denominator on the rhs of (4.7) has to be replaced by  $\cos \delta e^{+i\beta}$  and then this Möbius transformation no longer maps points of the unit circle onto points on the unit circle. In what follows we therefore restrict our considerations to real eigenvalues and since  $z_N = 1$ , all points  $z_n$  are located on the unit circle.

Since  $\det Q_\beta = -1$  for all  $\beta$ , the corresponding Möbius transformations are hyperbolic, i.e., have two fixed points, one of them attractive and the other repulsive. Using this fact and adopting the arguments of the literature,<sup>(19,22)</sup> one can now prove the existence of special frequencies (numbers that cannot be eigenvalues, or values, where the spectral density has to vanish): Consider the Möbius transformation  $R_{l\beta}$ . Its fixed points are given by

$$z_{\pm} = e^{\pm i\xi + i\beta(l+1)}, \quad e^{\pm i\xi} = \underbrace{\cos \beta \cos l\beta}_{=: \cos \xi} \pm i \underbrace{(1 - \cos^2 \beta \cos^2 l\beta)^{1/2}}_{=: \sin \xi} \quad (4.8)$$

where  $z_+$  is repulsive [ $|R'_{l\beta}(z_+)| > 1$ ] and  $z_-$  is attractive [ $|R'_{l\beta}(z_-)| < 1$ ]. Let  $\beta = \pi/L$ ; then  $L$  symmetric bond pairs in succession transform  $z_n$  into itself:  $z_{n-L} = e^{-2i\beta L} z_n = z_n$  if  $\theta_n = \theta_{n-1} = \dots = \theta_{n-(L-1)} = +1$ . Hence, the whole chain can be described by segments  $\mathbf{AS}^l$  ( $l=0, \dots, L-1$ ), i.e., one antisymmetric bond  $\mathbf{A}$  following  $l$  symmetric bonds  $\mathbf{S}$ . Therefore one has to satisfy

$$e^{2i\beta} = R_{l_1\beta} \circ \dots \circ R_{l_k\beta}(1), \quad \text{with } l_i \in \{1, \dots, L\} \quad (4.9)$$

But, as one can easily calculate for large  $L$ , the attractive fixed points  $z_-^j$  of  $R_{j\beta}$  are all within the segment  $\{e^{i\varphi} \mid \varphi \in [0, \pi/L]\}$  of the unit circle and the repulsive fixed points are located outside this segment. Thus the point  $z_n$  can never be fulfilled. In other words,  $\lambda = \cos \pi/L$  is no eigenvalue of  $\underline{A}$  and is therefore a special frequency.

Now choose  $\lambda$  close to the band edge, i.e.,  $0 < \beta \ll 1$ . Let  $\beta$  be no special frequency. As one can see from (4.7) in the case of an antisymmetric bond the variable  $z_n$  is mapped into the vicinity of the attractive fixed point of  $R_\beta$ , whose phase is a bit smaller than  $\beta$ . To satisfy boundary condition,  $z_0 = e^{2i\beta}$ , and this can only be achieved by a succession of symmetric bond pairs. The crucial point is that the Möbius transformation corresponding to a symmetric bond rotates  $z$  in the *clockwise* direction by an angle  $2\beta$  and therefore we must have  $2\pi/2\beta$  symmetric bonds in succession to reach  $z_0$ . But the probability of this event is  $p^{\pi\beta^{-1}}$ , i.e., very low if  $0 < \beta \ll 1$ . Since

$\lambda = 2 \cos \beta \approx 2 - \beta^2$ , it follows for the density of the purely real eigenvalues of  $\underline{A}$

$$\tilde{\rho}(\lambda) \approx \exp[-\pi |\log p| (2 - \lambda)^{-1/2}] \tag{4.10}$$

which is a typical singularity occurring in the spectral density of disordered binary alloys, also called Lifshitz singularities. We derived this singularity at the band edge  $\lambda = 2$  only for the purely real part of the spectrum. But note that

$$|M(t)| \leq e^{-t} \int_{-2}^{+2} dx e^{x^{1/2}} \rho_{\text{Re}}(x) \quad \text{with} \quad \rho_{\text{Re}}(x) = \int_{-2}^{+2} dy \rho(x + iy) \tag{4.11}$$

where  $\rho(x + iy)$  is the spectral density of  $\underline{A}$  (3.1). Hence the most important quantity is  $\rho_{\text{Re}}(x)$ , which we expect to be of the same form as the purely real part of the spectral density (4.10), as is indicated by the numerical diagonalizations of finite matrices (see Fig. 3).

An immediate consequence of such an essential singularity (4.10) of  $\rho_{\text{Re}}$  of  $\underline{A}$  at the point  $\lambda = 2$  is—via the Laplace transform occurring on the rhs of the inequality in (4.11)—the stretched exponential decay of the remanent magnetization, which we claimed above.

### 4.2. Moments of the Spectral Density

We return to Eq. (4.1) and expand the exponential function within the integrand. One observes that

$$M(t) = e^{-t} \sum_{v=0}^{\infty} \frac{a_{2v}}{(2v)!} \left(\frac{t}{2}\right)^{2v} \tag{4.12}$$

where

$$a_v = \int_{\mathbb{C}} d^2\lambda \lambda^v \rho(\lambda) \tag{4.13}$$

are the moments of the spectral density of the matrix  $\lambda$  [note that only the even moments of  $\rho(\lambda)$  do not vanish, as will be clarified in Appendix A]. The asymptotic behavior of  $M(t)$  is determined by the asymptotic behavior of the moments  $a_{2v}$  (i.e., the limit  $v \rightarrow \infty$ ). When we can show that the moments of the matrix  $\underline{A}$  do not grow faster with  $v$  than those of the corresponding diluted model, we have proved that  $M(t)$  decays asymptotically as the remanent magnetization of the corresponding diluted model, i.e., with stretched exponential. This will be shown below.

In Appendix A we derive the following equality:

$$a_{2\nu} = \sum_{i=1}^{\nu} \sum_{\substack{r_1, \dots, r_i \geq 1 \\ r_1 + \dots + r_i = \nu}} \mathcal{F}(r_1, \dots, r_i) \langle \theta^{r_1} \rangle \dots \langle \theta^{r_i} \rangle \quad (4.14)$$

where  $\theta$  is a random variable that is  $+1$  with probability  $p$  and  $-1$  with probability  $1 - p$ , i.e.,

$$\langle \theta^r \rangle = \begin{cases} 1 & \text{for } r \text{ even} \\ \eta & \text{for } r \text{ odd} \end{cases} \quad (4.15)$$

where  $\eta = 2p - 1$ . It is exactly this random variable that makes the difference to the moments of the spectral density of the corresponding diluted model, where  $\theta$  is  $+1$  with probability  $p$  and  $0$  with probability  $1 - p$ , representing the removed bond pairs in an otherwise ferromagnetic chain. In other words

$$\langle \theta_{\text{dil}}^r \rangle = p \begin{cases} < 1 \\ > \eta \end{cases} \quad (4.16)$$

Unfortunately,  $\langle \theta^r \rangle$  is sometimes greater and sometimes smaller than  $\langle \theta_{\text{dil}}^r \rangle$ , otherwise one could trivially write down an inequality for  $a_{2\nu}$ .

The factor  $\mathcal{F}(r_1, \dots, r_i)$  denotes the number of closed walks in one dimension with nearest neighbor steps only, which visit exactly  $i + 1$  different sites, start at any of them (and return to it), and jump  $r_1$  times back and forth between the leftmost site and its right neighbor,  $r_2$  times between the latter and its right neighbor, and so on. Its value is given by<sup>(23)</sup>

$$\mathcal{F}(r_1, \dots, r_i) = \frac{2\nu}{r_1} \binom{r_1 + r_2 - 1}{r_2} \binom{r_2 + r_3 - 1}{r_3} \dots \binom{r_{i-1} + r_i - 1}{r_i} \quad (4.17)$$

The rhs of (4.14) is a sum over all possible closed walks of length  $2\nu$ , each weighted by the product of several moments of the distribution of the random variable  $\theta$ . The value of this product is given by  $\eta^{\mathcal{N}}$ , where  $\mathcal{N} = \#\{r_j | r_j \text{ is odd}\}$ . If the frequencies (of jumping forth and back)  $r_j$  are large, it is not important for the order of magnitude of  $\mathcal{F}(r_1, \dots, r_i)$  how many frequencies are odd. In this case one can think of frequencies  $r_j$  to be even or odd both with probability one-half and this does not have much influence on the combinatorial factor (4.17). Suppose therefore that

$$a_{2\nu} \approx \sum_{i=1}^{\nu} \left\{ \sum_{\substack{r_1, \dots, r_i \geq 1 \\ r_1 + \dots + r_i = \nu}} \underbrace{\sum_{\mathcal{N}=0}^i \binom{i}{\mathcal{N}} \left(\frac{1}{2}\right)^i \eta^{\mathcal{N}} \cdot \mathcal{F}(r_1, \dots, r_i)}_{\leftrightarrow \langle \theta^{r_1} \rangle \dots \langle \theta^{r_i} \rangle} \right\} \quad (4.18)$$

where the second sum within the brackets has replaced (as indicated) the product  $\langle \theta^{r_1} \rangle \dots \langle \theta^{r_i} \rangle$  on the rhs of (4.14) and is just the sum over all possible choices of  $\mathcal{N}$  odd frequencies out of  $i$  frequencies, multiplied by their probabilities  $(1/2)^i$  and their weight  $\eta^{-\mathcal{N}}$ . But this sum is equal to  $p^i$  and hence the rhs of (4.18) is identical to the expression for the moments of the corresponding diluted model.

Let us discuss the underlying assumptions of the last conclusion: The property of the frequencies to be even or odd is certainly not random when the diameter  $i$  of the walk is as big as its length  $v$ . Because the walk has to return to its origin *and* has to go far away, it can mostly jump only one time forth and back between two neighboring sites: forth when it is on the way to its right or left end—or turning point—and back when it returns. This means that  $r_j$  has to be equal to one—which is an odd number—very often and hence for  $i \approx v$  the weight  $p^i$  of the combinatorial factor is too large, since this weight originates in making no difference between odd and even frequencies. Furthermore, it is always  $\mathcal{N} \leq \min\{i, v - i\} \leq \lfloor v/2 \rfloor$  because the smallest even number greater than one is two and  $r_1 + \dots + r_i = v$  has to be fulfilled. The last two objections would imply that the approximation (4.18) gives an upper bound for  $a_{2v}$  and therefore the remanent magnetization of the diluted model gives an upper bound for the asymmetric model.

Finally, we counted numerically (for fixed  $i$  and  $v$ ) the number  $n_{i,v}(\mathcal{E})$  of walks with  $\mathcal{E} (\mathcal{E} = i - \mathcal{N})$  even frequencies  $r_j$  and got the histograms depicted in Fig. 4, where in Fig. 4b we chose  $v = 24$  and different values for  $i$ , and in Fig. 4a we chose  $i = v/2$ ,  $v = 12, 24, 32$ . All results for  $n_{i,v}(\mathcal{E})$  can be fitted by a Gaussian distribution, the points in Fig. 4a by

$$n_{i,2i}(\mathcal{E}) = \frac{2a_i}{(2\pi\sigma_i^2)^{1/2}} \exp - \frac{(\mathcal{E} - x_i)^2}{2\sigma_i^2} \tag{4.19}$$

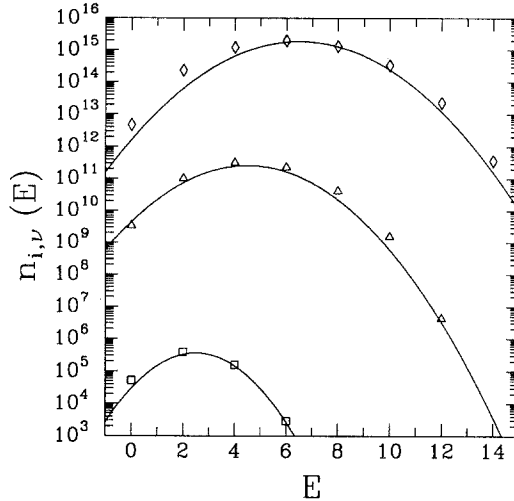
where the factor 2 is due to the fact that  $n_{i,v}(\mathcal{E}) = 0$  for  $\mathcal{E}$  odd if  $v$  and  $i$  are even. We get

$$\begin{aligned} a_{6,12,16} &= 5 \cdot 10^5, 5 \cdot 10^{11}, 4 \cdot 10^{15} \\ x_{6,12,16} &= 2.5, 4.5, 6.5 \\ \sigma_{6,12,16}^2 &= 1.25, 2.5, 3 \end{aligned} \tag{4.20}$$

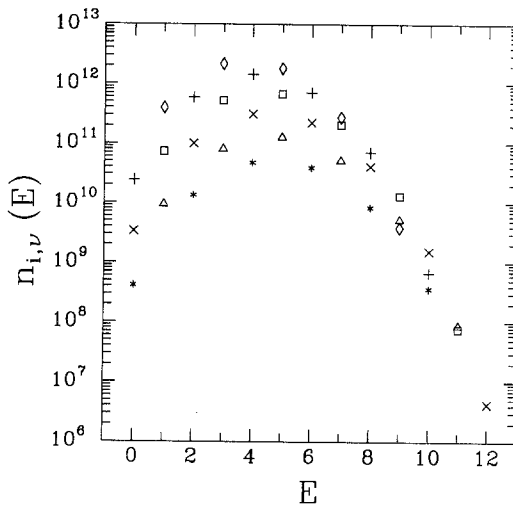
This means that the peak of  $n_{i,2i}(\mathcal{E})$  is located at a value  $x_i$  that is a little bit smaller than  $i/2$  and the variance of  $n_{i,2i}(\mathcal{E})$  grows linearly with  $i$ . This is exactly what we expect if the property of the frequencies  $r_j$  being even is random.

Finally, let us mention the fact that the fully asymmetric case (by which we denote the case  $p = 0.5$  or  $\eta = 0$ ) leads to an interesting com-

binatorial problem of counting all closed, one-dimensional walks that are restricted to jump always an even number of times forth *and* back between neighboring sites. This is due to the fact that in this special case  $\langle \theta^r \rangle$  is equal to zero for  $r$  odd, and is equal to unity for  $r$  even, so that only the above-mentioned walks contribute to the sum on the rhs of (4.14).



(a)



(b)

Fig. 4. (a) The number  $n_{i,\nu}(\mathcal{E})$  of walks with  $\mathcal{E}$  even frequencies  $r_1, \dots, r_i$  ( $\nu=12, 24, 32$ ,  $i=\nu/2$ ) counted numerically. We fitted with the Gauss function (4.19) and obtained the parameters (4.20). (b) The same as in (a), but for  $\nu=24$  and different values of  $i$ ; from top to bottom,  $i=9, 10, 11, 12, 13, 14$ .

## 5. NUMERICAL SIMULATIONS

Now we want to test the theoretical predictions made in the preceding sections. To simulate the zero-temperature dynamics of the asymmetric spin chain (with discrete couplings) we used multispin coding techniques and parallel updating. Each spin has two neighbors, so that there are altogether four different configurations. Two of them yield a vanishing local field. Hence in approximately one-half of all cases the field acting upon the spin is zero and we have to generate a random number to flip the spin with probability one-half. To avoid **if** structures, which make every multispin coding inefficient, and to produce random bits very fast we used the following trick.<sup>(24)</sup> The main loop of the program, containing the update process, is

```

do 100 i = 1, ll
    ir(i) = ir(i - 250).xor.ir(i - 103)
    nh = n(i - 1).eqv.jj(i)
    m(i) = (ir(i).and.(n(i + 1).xor.nh)).or.(n(i + 1).and.nh)
100 continue

```

(5.1)

The state of the individual spins is stored in the array **n** of length **ll**. Every bit of a computer word **n(i)** corresponds to one spin, so that **n(i)** contains 32 (or 64) spins and the actual length of the chain is equal to  $32 * ll$  (or  $64 * ll$ ). Analogously, the couplings  $J_{i,i-1}$  are stored within the array **jj**. The signs of the couplings  $J_{i+1,i}$  are gauged away, as was done in (3.3)–(3.4), so that they are all positive. The array **ir** is an array for the Kirkpatrick–Stoll random number generator<sup>(25)</sup>: if the elements **ir(-249)** to **ir(0)** are computer words whose bits are one and zero randomly with probability one-half, then this rule produces a new word with random bits. After the loop is completed, the words **ir(ll-249)** to **ir(ll)** are shuffled to **ir(-249)** to **ir(0)**.

The variable **nh** contains the contribution of the left neighbors of the spins to their local field (i.e., a bit of **nh** is one iff  $\sigma_{i-1}$  and  $J_{i,i-1}$  have the same sign and it is zero otherwise). Consider now the last line of the loop, where **m(i)** contains the new values of the spins. The local field acting upon a spin is zero, if the corresponding bit in **n(i + 1).xor.nh** is one, because in that case  $\sigma_{i+1}$  and  $\sigma_{i-1} \cdot J_{i,i-1}$  have different signs. Exactly in this case the spin has to take over one random bit in **ir(i)**. If the field is not zero [i.e., the corresponding bits in **n(i + 1)** and **nh** are both one or both zero, and the **xor** operation is false], the bit has to take over the value of the operation **n(i + 1).and.nh**.

Using this loop, we reached on a sparc workstation a speed of  $10^7$  spin



updates per second and for a typical run with  $10^7$  spins and  $t = 10^5$  sweeps through the whole chain we needed 30 hours CPU time. On a Cray YMP one could expect a speedup of a factor 100, i.e.,  $10^9$  spin updates per second (note that the above main loop can be completely vectorized). First we compared the decay of the remanent magnetization of the asymmetric spin chain with that of the corresponding diluted spin chain, where we replaced each antisymmetric bond occurring with probability  $1 - p$  in the former by a removed bond pair in the latter [to simulate this we have of course to modify the update rule in (5.1)]. In Fig. 5 we show the result for  $p = 0.9$  and immediately observe that  $M(t)$  of the asymmetric chain is always lower than that of the diluted chain, which decays with a stretched exponential (in fact  $M(t) = \exp[-(t/300)^{1/2}]/\sqrt{t}$  yields a very good fit for the latter, indicating that the asymptotic regime, where the exponent should be  $1/3$  instead of  $1/2$ , is still not reached). Furthermore, one sees for long times enormous fluctuations in the remanent magnetization of the asymmetric spin chain, whereas the diluted chain shows a much smoother behavior. This is due to the fact that in the latter for long times only the long segments relax; the shorter have reached a steady state, where the spin do not flip any more. In contrast, the antisymmetric bond pairs give rise to chaotic spin flips at all times, as described in Section 3.

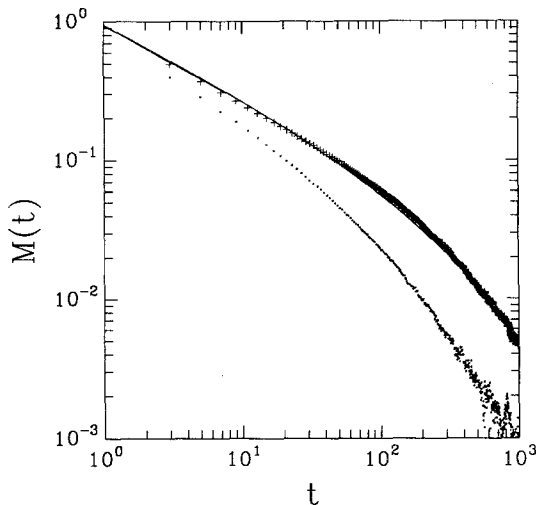


Fig. 5. Comparison of the decay of the remanent magnetization of the asymmetric spin chain (lower points) and the corresponding diluted chain (upper points) for  $p = 0.9$  obtained via numerical simulation. The number of spins is  $N = 10^7$ . The full curve is the fit mentioned in the text.

In Fig. 6 we show how the remanent magnetization approaches the algebraic decay ( $1/\sqrt{t}$ ) for  $p \rightarrow 1$ , by performing simulations for  $p = 0.9$ , 0.99, and 0.995. There is always a characteristic time up to which the decay is nearly algebraic, but for greater times the decay gets faster. This is in full correspondence to the considerations of Section 3. To detect the correct relaxation behavior is difficult even in the diluted case, as we mentioned in connection with Fig. 5; nevertheless, we tried to extract the exponent  $\nu$  under the assumption that the decay is of the form  $\exp[-(t/\tau)^\nu]$  in Fig. 7 and found  $\nu = 0.25 \pm 0.10$ .

Finally, we performed simulations of the same model in two dimensions. This means we considered a two-dimensional square lattice instead of a one-dimensional spin and coupled nearest neighbor spins by  $\pm J$  interactions, i.e., the sum over  $j$  on the rhs of the definition of the local field (2.1) is now over four nearest neighbor spins. In contrast to the one-dimensional case, which is equivalent to a ferromagnet in the symmetric case, one can now introduce the asymmetry in different ways. First one can think of a two-dimensional ferromagnet and introduce with probability  $1-p$  antisymmetric bond pairs for the ferromagnetic, symmetric bond pairs. Second one can think of a  $\pm J$  spin-glass model in two dimensions and do there the same thing. In both cases we got an exponentially rapidly

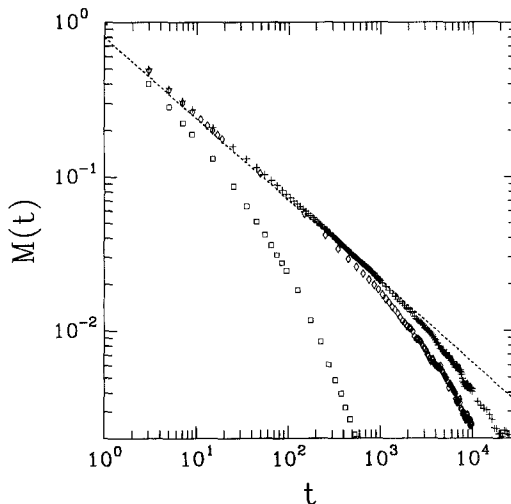


Fig. 6. A log-log plot of the remanent magnetization of the asymmetric spin chain for different values of  $p$ : From top to bottom,  $p = 0.995$ ,  $p = 0.99$ ,  $p = 0.9$ . The points for larger times are averaged over certain time intervals.  $N$  is  $10^7$  and for each value of  $p$  we averaged over 100 configurations of the couplings. The straight line is the curve  $1/\sqrt{t}$ , which is the decay for  $p = 1$ , i.e., the symmetric case.

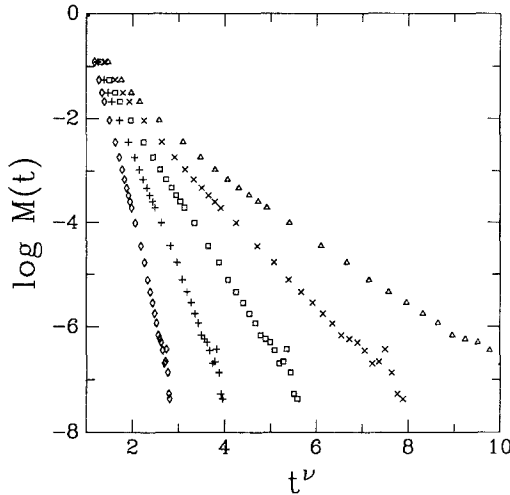


Fig. 7. The logarithm of the remanent magnetization of the asymmetric spin chain for  $p = 0.9$  plotted against  $t^\nu$  for different  $\nu$  (from top to bottom, 0.35, 0.30, 0.25, 0.20, 0.15). If the decay is stretched exponential with exponent  $\nu$ , the plot should give a straight line. One observes the different bendings of the two outermost curves  $\nu = 0.15$  and  $\nu = 0.35$ .

decaying remanent magnetization (see Fig. 8) starting with a random initial state. The difference lays in the fact that the remanent magnetization of the two-dimensional spin-glass model decays completely to zero even in the symmetric case, whereas in the ferromagnet with asymmetric interactions it decays to a nonvanishing if  $p$  is close enough to one.

This can be understood as follows<sup>(26)</sup>: Consider a part of the system that is fully magnetized, e.g., all spins up in a certain region of the lattice. Suppose that in the midst of such a region only one antisymmetric bond connects two neighbors. If both would have only two neighbors, as in the one-dimensional case, one of the spins would certainly flip because it has zero field. But in two dimensions there are four neighbors, and hence the other three neighbors stabilize the spin against the influence of the spin by which it is connected with an antisymmetric bond pair. Thus, in a ferromagnet with asymmetry the remanent magnetization does not need to decay completely if only the concentration of noise-producing antisymmetric bond pairs is small enough. This is different in the spin-glass case. Because of the frustrations caused by the random distribution of ferro- and antiferromagnetic bond pairs, the local fields vanish very frequently. This gives rise to noise, even for symmetric couplings, and the remanent magnetization decays completely.

Hence we conclude that in two dimensions there is no indication for a similar asymmetry-induced transition in the  $\pm J$  spin glass as observed in

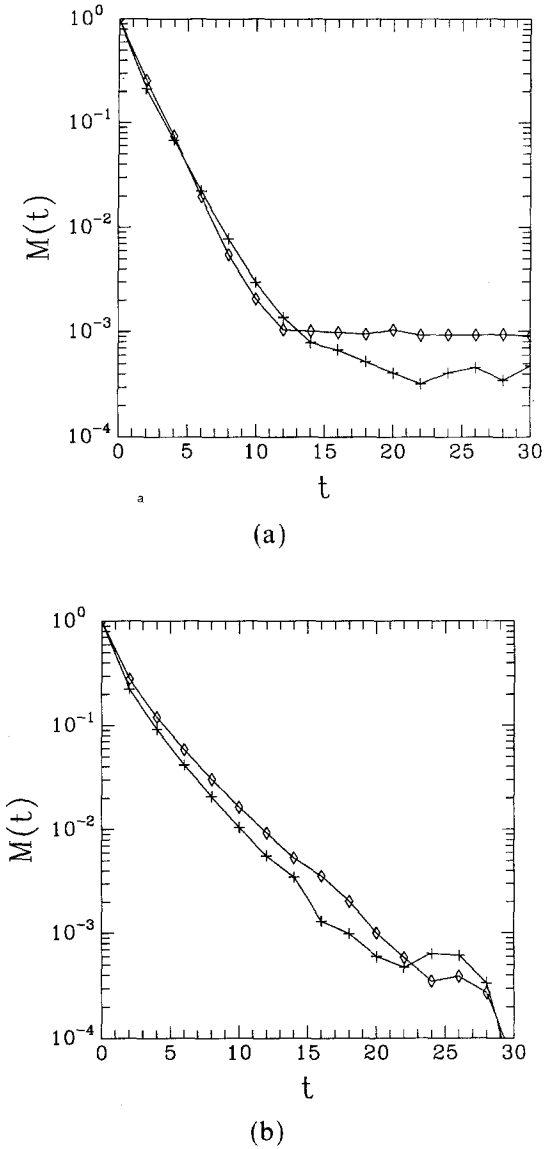


Fig. 8. The remanent magnetization of the asymmetric  $\pm J$  spin glass in two dimensions (the y axis is logarithmic) containing antisymmetric bonds occurring with probability  $p$ . The size of the systems is  $1000 \times 1000$  (i.e.,  $N = 10^6$ ; the individual curves are averaged over 100 runs. (a) The ferromagnet for  $p = 1$  (squares) and  $p = 0.9$  (crosses), (b) the spin glass for  $p = 1$  (squares) and  $p = 0.9$  (crosses).

the one-dimensional case. We conjecture that this is also true for higher dimensions. Since there is analytical<sup>(8)</sup> and numerical<sup>(9,10)</sup> evidence for a transition in the mean field (i.e., infinite dimensional) version of this model, one can speculate whether there is a critical dimension above which this transition exists or not.

## 6. CONCLUSION AND OUTLOOK

We have demonstrated that the  $\pm J$ -spin chain shows at zero temperature a drastic change in its dynamics if asymmetry is introduced. This change is manifested by the decay of the remanent magnetization, which is algebraic for purely symmetric couplings and becomes stretched exponential for any nonvanishing concentration of antisymmetric bond pairs in the infinite chain. The underlying picture is that of finite, purely symmetric segments, bounded by spins, which are coupled by an antisymmetric bond pair and therefore generate random noise. The chaotic flips of these spins produce kinks that move as a one-dimensional random walk through the finite segments and destroy the remanent magnetization. This leads to the similarity between the diluted (symmetric) spin chain and the model considered here.

The dynamics of the asymmetric  $\pm J$ -spin chain is extremely difficult to investigate analytically, mainly due to the occurrence of complex eigenvalues within the spectrum of the relaxation matrices. Although we are far from a complete understanding of this spectrum, which shows fascinating properties, we were able to prove the existence of special frequencies and detected a Lifshitz singularity at the band edge leading to stretched exponential decay. We think that it is worth to investigate further the spectrum of non-Hermitian (tridiagonal) matrices in the future.

It is also interesting to consider other quantities than the remanent magnetization. As we mentioned earlier, we expect the dynamical decay for other correlation functions to remain the same. But if one looks, for instance, at the stationary state of spatial correlations among spins at different sites, then even in the case of only one impurity one is confronted with a system of linear equation that we are unable to solve analytically by standard methods. Furthermore, it would be very interesting to calculate the size dependence of the entropy or even its increase with time in this simple model.<sup>(26)</sup> Again one is confronted with tremendous analytical difficulties. Nevertheless, both cases can easily be investigated with numerical methods and we think that the simple, abstract model introduced in this paper is a rich but still unexplored source of insight into the non-equilibrium physics of the systems mentioned in the introduction.

The Glauber dynamics of the asymmetric  $\pm J$  chain considered here can also be interpreted in a different way, yielding a connection to a whole class of similar models. The main point is the definition of the update process in the case when the local field is exactly zero. This can only happen in the  $\pm J$  chain. In the above definition random numbers were chosen at every timestep and for every spin with zero field separately to fix the new state of these spins. This is equivalent to a model with annealed random fields with a symmetric distribution function, which satisfies the two conditions

$$\text{prob}\{h_i = 0\} = 0 \quad \text{and} \quad \text{prob}\{|h_i| > J\} = 0 \quad (6.1)$$

i.e., no  $\delta$ -function at  $h=0$  and the absolute value of  $h$  is bounded from above by  $J$ . The effect of these random fields is noise and therefore a vanishing remanent magnetization even at  $T=0$ . Another way of interpretation is that a spin with vanishing local field chooses at random one of its two neighbors with probability  $1/2$  and takes into account only the local field caused by this neighbor.

The situation changes drastically if instead of the annealing a quenched randomness is introduced. This can be achieved by choosing a quenched random field which remains the same for all times, or by choosing one of the two neighbors of each spin once at the beginning of the dynamical process and having the local field consist only of the contribution of this neighbor. In both cases the remanent magnetization is no longer zero and the relaxation is exponential. A detailed analysis with analytical results for these models will be given elsewhere.<sup>(12)</sup>

There are essentially two known exact results for 1D spin-glass dynamics: In ref. 28 a model with random infinite fields  $h_i = -\infty, 0, +\infty$  is investigated. In fact this model can be mapped exactly<sup>(12,15)</sup> on the trapping problem in 1D<sup>(14)</sup> and one can write down directly the results for the remanent magnetization (2.18). In this case the distribution of the fields does not satisfy both conditions (6.1) and the field is zero or dominates the contributions of the two neighbors to the local field. At  $T=0$  the fields only need to be larger than  $2J$  to give the same result.

The other known result<sup>(27)</sup> is the calculation of the remanent magnetization of the symmetric 1D chain with a *continuous* distribution of the couplings. The result of  $1/3$  at  $T=0$  is independent of the details of the distribution. The system relaxes exponentially into this state. At small but finite temperatures a very slow decay of the remanent magnetization can be observed. This model can also be generalized<sup>(12)</sup> to include asymmetric couplings and the fully asymmetric variant of this model is identical to the one with quenched favorite neighbors mentioned above.

The methods developed in ref. 12 can also be applied to spin glasses on Cayley trees with any branching number, but rigorous results are available only in the symmetric case. It would be interesting to see how in higher dimensions the behavior will evolve with respect to the SK model. But this can only be analyzed numerically because analytical results are not available.

**APPENDIX A**

Here we calculate the value of the periodic continued fraction (3.15),  $S(\lambda) = \lim_{N \rightarrow \infty} S^{(N)}(\lambda)$ . Consider the map (in the variable  $\omega$ )

$$\tilde{S}_\lambda(\omega) := \frac{x_1}{1 - \frac{x_2}{1 - \frac{x_3}{1 - \dots - \frac{x_k}{1 + \omega}}}}$$

with  $x_1 = \dots = x_k = x^2 := \lambda^{-2}$  (A.1)

Then  $S(\lambda)/\lambda$  is identical to the stable fixed point  $\omega_{\text{fix}}$  of the map  $\tilde{S}_\lambda$ ,<sup>(17)</sup> i.e.,

$$S(\lambda) = \lambda \omega_{\text{fix}}, \quad \tilde{S}_\lambda(\omega_{\text{fix}}) = \omega_{\text{fix}}$$
 (A.2)

To calculate the fixed point, one observes that

$$\tilde{S}_\lambda(\omega) = \frac{A_{k-1}\omega + A_k}{B_{k-1}\omega + B_k}$$
 (A.3)

where

$$\begin{aligned} A_{p+1} &= A_p - x^2 A_{p-1}; & A_{-1} &= -1, & A_0 &= 0 \\ B_{p+1} &= B_p - x^2 B_{p-1}; & B_{-1} &= 0, & B_0 &= 1 \end{aligned}$$
 (A.4)

It follows that  $A_p = x^2(\rho_+^p - \rho_-^p)/(\rho_+ - \rho_-)$  and  $B_p = x^2(\rho_+^{p+1} - \rho_-^{p+1})/(\rho_+ - \rho_-)$ , with  $\rho_\pm = 1/2 \pm (1/4 - x^2)^{1/2}$  and therefore

$$\tilde{S}_\lambda(\omega) = x^2 \frac{(\rho_+^{k-1} - \rho_-^{k-1})\omega + (\rho_+^k - \rho_-^k)}{(\rho_+^k - \rho_-^k)\omega + (\rho_+^{k+1} - \rho_-^{k+1})}$$
 (A.5)

Thus the fixed point equation (A.2) to be solved is a quadratic equation, whose solution is given in the text, (3.17).

## APPENDIX B

Here we derive the formula for the moments  $a_\nu$  [Eq. (4.14)] of the spectral density  $\rho$  of the matrix  $\underline{A}$ , (3.1). First we observe that

$$a_\nu = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N \lambda_i^\nu = \lim_{N \rightarrow \infty} \frac{1}{N} \text{Tr} \underline{A}^\nu \quad (\text{B.1})$$

where  $\lambda_1, \dots, \lambda_N$  are the eigenvalues of the  $N \times N$  matrix  $\underline{A}$ . But for any matrix  $\underline{A} = (a_{ij})$  one has

$$\text{Tr} \underline{A}^\nu = \sum_{p_1, \dots, p_\nu \in \{1, \dots, N\}} a_{p_1 p_2} a_{p_2 p_3} \cdots a_{p_\nu p_1} \quad (\text{B.2})$$

Since only  $A_{i, i+1} \neq 0$  and all other elements of  $\underline{A}$  vanish, the rhs of (B.2) can be viewed as a sum over all possible closed walks of length  $\nu$  in one dimension with nearest neighbor jumps only. One starts with an arbitrary matrix element, e.g.,  $A_{i, i+1}$ , and jumps either to the right, i.e.,  $A_{i+1, i+2}$ , or to the left, i.e.,  $A_{i+1, i}$ . Since one cannot return to the origin with an odd number of steps (note that the diagonal elements are zero), all odd moments vanish. Because one has to return to the starting element with each element  $A_{i, i+1}$  also the corresponding element  $A_{i+1, i}$  has to occur in the product. Therefore a walk of length  $2\nu$  gives rise to a product  $\theta_{i_1} \cdots \theta_{i_\nu}$ , where  $\theta_i = A_{i, i+1} A_{i+1, i}$  and  $i_{l+1} = i_l \pm 1$ .

The value of the products depends only upon the frequencies with which certain bond pairs are visited. This observation leads us to the introduction of the combinatorial factor  $\mathcal{F}(r_1, \dots, r_l)^{(23)}$  with  $r_1 + \dots + r_l = \nu$ , which counts the number of walks characterized by the frequencies  $r_1, \dots, r_l$ : the leftmost bond pair is visited  $r_1$  times, the next bond pair (to its right) is visited  $r_2$  times, etc., and altogether  $l$  different bond pairs are visited. All these walks lead to weighting factors of the form

$$\theta_{i_1}^{r_1} \cdots \theta_{i_l}^{r_l} \quad (\text{B.3})$$

but the latter still depends on the starting point as well as on the number of frequencies to the right of the starting point.

Now the self-averaging property of (B.1) comes into play: (B.1) is an average over all starting points and (in the limit  $N \rightarrow \infty$ ) is identical to an average over all possible configurations of the random variables  $\theta_{i_j}$  ( $j = 1, \dots, l$ ) in (B.3). Hence, the weighting factor (B.3) of all walks characterized by the frequencies  $r_1, \dots, r_l$  becomes  $\langle \theta^{r_1} \rangle \cdots \langle \theta^{r_l} \rangle$ . Thus we end with

$$\frac{1}{N} \text{Tr} \underline{A}^\nu \xrightarrow{N \rightarrow \infty} \sum_{l=1}^{\nu} \sum_{\substack{r_1, \dots, r_l \geq 1 \\ r_1 + \dots + r_l = \nu}} \mathcal{F}(r_1, \dots, r_l) \langle \theta^{r_1} \rangle \cdots \langle \theta^{r_l} \rangle \quad (\text{B.4})$$

The value of the combinatorial factor is given in the text.



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